# **II. Hyperbolic equations**

We will solve the vibrating string equation. We will consider the following cases:

* Cauchy problem
* First boundary problem
* Second boundary problem
* Non-homogeneous equation

## **4. Movement of the unlimited string**

We analyze the movement of the unbounded string, which is described by the Cauchy problem for the vibrating string equation. Using D'Alembert method, we transform the given equation to the canonic form. After integration of the result, we obtain the general solution of the vibrating string equation. Using the initial conditions, we find the solution of the Cauchy problem, which is called the D'Alembert formula. Running wave phenomenon is considered as an application.

### **4.1. Problem statement**

We would like to analyze the movement of the long enough string. Then we suppose that the string is unlimited. The phenomenon is described by the ***string vibrating equation***

 *utt = a*2*uxx* , (4.1)

where the spatial variable *x* changes from -∞ to ∞.

We know that the differential equations are solved with additional conditions. We cannot to add the boundary conditions, because we do not have the boundary ends. However, we can have the initial conditions. We know that the second order ordinary differential equations are solved with two initial conditions (initial state and initial velocity). Suppose the initial state *ϕ* of the string and its initial velocity *ψ* are given. Of course, these values can depends from the spatial variable *x.* Let the initial time be 0. Therefore, we have the initial conditions

 *u*(*x*,0) = *ϕ*(*x*), *ut*(*x*,0) = *ψ*(*x*), -∞ < *x* < ∞. (4.2)

Thus, we have the problem (4.1), (4.2). This is called ***Cauchy problem*** as the analogical problem for the ordinary differential equations.

### **4.2. Canonic form of the equation**

We solve our problem with using the transformation of the equation to the canonic form. This is the ***D'Alembert method***. Let us have the equation



Its analysis starts by the calculation of the discriminant



For our case, the variable *y* is *t.* Then we have the value of the parameters



Then we determine *D = a*2. Therefore, our equation is hyperbolic.

We know that for the hyperbolic case one consider two characteristic equations

 

 

For our case, we determine the equations

  (4.3)

  (4.4)

These equalities can be transformed



The general solution of the first equation is *x-at = c.* The general solution of the second equation is *x+at = c.*

By the known theory, for the transformation of the given equation to the canonic form it is necessary to use the following variables

 *ξ = x-at*, *η = x+at.* (4.5)

After exchange of the variables, we obtain the equation



where



For our case the function  is zero, because *F=*0 for the equation (4.1). Find the new coefficients using the derivatives



Now we find



Then we have the equation

  (4.6)

This is the ***canonic form*** of the vibrating string equation.

### **4.3. General solution of the vibrating string equation**

Find the solution of the equation (4.6). We have the equality



The derivative of a function of one variable is zero whenever this is a constant. However, the partial derivative of the function of two variables with respect to its first argument is zero whenever this function does not depend of this argument. Therefore, it can be the arbitrary function of the second argument. Thus, from the equality (4.6) it follows that

  (4.7)

where the function *f* is arbitrary.

Now it is necessary to solve the equation (4.7). If the derivative of the function is another function, then the initial function is equal to the integral of the second function plus an arbitrary constant. This is true for the functions of one variable. However, for the functions of two variable the result will be equal to the integral of the second function with respect to variable of differentiation plus an arbitrary function of another variable. Thus, after integration of the equality (4.7) by *ξ*, we get the equality



where the function *g* is arbitrary. The first term at the right hand-side of this equality is a function of the variable *ξ*. Denote this function by *h.* Then we obtain



Return to the initial variables *x* and *t.* Using the formulas (4.5), we find

  (4.8)

The formula (4.8) give the ***general solution*** of the vibrating string equation (4.1). This depends from two arbitrary functions of one variable.

### **4.4. D'Alembert formula**

We would like to determine the solution of the Cauchy problem for the vibrating string equation. Therefore, it is necessary to find the partial solution of this equation that satisfies the initial conditions (4.2).

Using the first initial condition, we get



Determine the derivative of the function *u* with respect to the time. We have



Using the second initial condition, we obtain



Integrate this equality by *x* from a fixed value *x*0 to the arbitrary value *x*. We get



Denote the value  by *c.* We obtain the equality



Now we have the system of two linear algebraic equations

  (4.9)

with respect to the numbers  and . Summing these equalities, we find



Differing these equalities, we obtain

 

Put the result to the formula (4.8). We determine



Thus, we find

  (4.10)

Thus we found the solution of the Cauchy problem (4.1), (4.2). This is called the ***D'Alembert formula***.

Remark. The ordinary differential equation of *n* order has a general solution that depends from *n* unknown constants. The partial differential equation of *n* order with respect to *m* variables has a general solution that depends from *n* unknown functions of *m*–1 variables. Particularly, the vibrating string equation is the second order equation with respect to the function of two variables. Its general solution depends from two arbitrary functions of one variable.

### **4.5. Travelling waves**

Consider the concrete case of the Cauchy problem (4.1), (4.2). Suppose *a =* 1, the function *ψ* is zero, and the function *ϕ* is determined by the Figure 4.1.



Figure 4.1. Initial position of the string.

Using the D'Alembert formula, we get



Determine the position of the string at the following time:

*t*0 = 0, *t*1 = 1/4, *t*2 = 1/2, *t*3 = 3/4, *t*4 = 1, *t*5 = 5/4.

Note that the position of the string at the time is the superposition of the two semiwave  that is the function  with the left shift by *t* and  that is the function  with the right shift by *t.*

We have the following results (see Figure 4.2). This phenomenon is called the ***running wave***.



Figure 4.2. Running waves.

By the way, the influence of the parameter *a* is clear. This is the velocity of the wave. If this parameter increases, then the running away of the wave increases too.

### **Conclusions**

* The movement of the unbounded string is described by the Cauchy problem for the vibrating string equation.
* The D'Alembert method of solving of the vibrating string equation is based on its reduction to the canonic form.
* The general solution of the vibrating string equation is obtained by the direct integration of its canonic form.
* The solution of the Cauchy problem for the vibrating string equation is determined by the D'Alembert formula.
* The travelling wave phenomenon can be analyzed as an application of the D'Alembert formula.

### **Task. Travelling waves**

Consider theCauchy problem for the vibrating string equation

*utt = uxx* , -∞ < *x* < ∞, *t* > 0;

*u*(*x*,0) = *ϕ*(*x*), *ut*(*x*,0) = 0.

The function *ϕ* is given, see the following images:



 Variant 1 Variant 2 Variant 3



 Variant 4 Variant 5 Variant 6



 Variant 7 Variant 8

**Actions**

It is necessary perform the following steps:

1. Write the complete problem statement.
2. Write the D'Alembert formula for the given case.
3. Show the graphs of travelling waves such that all possible forms of the string will be determined (for example of the lecture, we have 6 different forms).

Use the example from the lecture as the sample.